2015WMTC Advanced Level

Team Round Problems

1. Consider a sequence $\{a_n\}$ in which $a_1=3$ and $a_{n+1}=\dfrac{1+a_n}{1-a_n}$($n$ is positive natural numbers). Find a_{2016}.

2. We call a 3-digit number $\overline{abc}$ an "arithmetic sequence number" if $a+c=2b$. How many "arithmetic sequence numbers" are there?

3. Let x be a non-zero natural number and let x^2-4, $2x$, and $x+1$ be the edge lengths of a triangle. Find all the possible perimeters of this triangle.

4. Let M be a non-empty set of integers. $k\in M$ is called an "isolated" element in M if both $k-1\notin M$ and $k+1\notin M$. Suppose $M=\{1, 2, 3, 4, 5\}$. How many subsets of M contain only one "isolated" element?

5. Find the maximum value for function $f(x)=2\sqrt{3}\sin 2x+4\sin x+8\sqrt{3}\cos x$.

6. Suppose a set contains n integers with half even numbers and the other half odd numbers. If P is the probability when the sum of any two randomly selected numbers from this set is even, find the largest positive integer n so that $P\leqslant\dfrac{99}{200}$.

7. Find the minimum value for function $y=3x^2-7x+2-2\sqrt{2x^2-3x-1}$.

8. Suppose α is a real root of $x^3+x-4=0$ and β is a real root of $x+\sqrt[3]{x}-4=0$. Find the value of $(\alpha+\beta)$.

9. Let $\{a_n\}$ and $\{b_n\}$ be two arithmetic (equal difference) sequences and let S_n and T_n be the corresponding sums of their first n terms, respectively. If $\dfrac{2S_n}{3T_n}=\dfrac{4n+19}{2n+2}$, find the number of prime numbers that are in the form of $\dfrac{8a_n}{3b_n}$.

10. Let circles O and K be the major circle and one of the minor circles for sphere O and that their common chord AB has the same length as the radius of sphere O as shown in figure 1, and $OK=\dfrac{3}{2}$. If the dihedral angle formed by two planes that contain circle O and circle K is $60°$, find the surface area of the cube that is "circumscribed" by the sphere O.

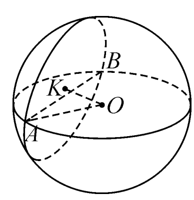


Fig.1

11. Let a and b be real numbers and $b \neq 1$. If the three terms a^2b, $ab^{\sin 2x}$, and a form both an arithmetic (equal difference) sequence and a geometric (equal proportion) sequence, find the value of $\cos\left(4x - \frac{\pi}{3}\right)$.

12. If $a > 3b > 0$, find the maximum value for $\frac{3a + 8a^2b - 16ab^2}{8b^2 - 4ab}$.

13. Suppose the straight line $x = m$ $(1 \leqslant m \leqslant 3)$ intersects functions $f(x) = \log_2(x+1) + 1$ and $g(x) = 4 - 2^{x-1}$ at points M and N, respectively. Find the range of values of $|MN|$.

14. Given function $g(x) = ||x-1|-1|$ $(x \geqslant 0)$. If function $f(x) = -\sin\frac{\pi}{2}x$ $(x \leqslant 0)$ has exactly n points that are the symmetric images of points from $g(x)$ reflect over the y-axis, find the value of n.

15. If a sequence $\{a_n\}$ satisfies $a_{n+1} + (-1)^n a_n = n$ (n is positive natural number) and the sum of its first n terms is $S_n = 2550$, find n.

16. Given a triangle $\triangle ABC$. Suppose $AB = 2$, $BC = 4$, and BD, the angle bisector of $\angle ABC$ has the length of $\frac{4\sqrt{2}}{3}$. Find the radius of the inscribed circle of $\triangle ABC$.

17. Given that a, b, and c are positive natural numbers where a is prime, $3b - 1$ is composite, both b and $2ab$ are perfect squares, and $8a + 2b + 3c \leqslant 45$. Find the sum of all possible values of abc.

18. Suppose in $\triangle ABC$, $\tan B = 4\tan C$ and $b^2 - c^2 = 12$. Find the area of this triangle's inscribed circle when its circumscribed circle is the smallest.

19. As shown in figure 2, B is the midpoint of the tangent PA to circle O, A is the point of tangency, C is a point on PA, line segments PDE, BFE, and CGE are all secant lines of circle O, and $\angle FAG = \angle FPB$. If $PA = EC = 6$ and $ED = 5$, find the length of EB.

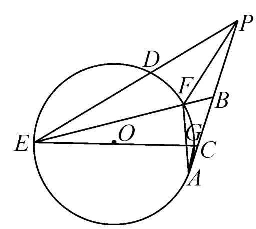


Fig.2

20. Let $V\text{-}ABC$ be a triangular pyramid with E, F, G, and H be the midpoints of edges AB, AC, VB, and VC, respectively. If α is the angle between planes AGH and VEF, find $\sin\alpha$.

Team Round Answers

1. $\frac{1}{2}$.

2. 45.

3. 25, 15.

4. 13.

5. 17.

6. 100.

7. -2.

8. 4.

9. 3.

10. 32.

11. $\frac{1}{2}$.

12. $-4\sqrt{3}$.

13. $[0, 3]$.

14. 3.

15. 100.

16. $3 - \sqrt{5}$.

17. 108.

18. $3 - \sqrt{5}$.

19. $4\sqrt{3}$.

20. $\frac{6\sqrt{2}}{11}$.

Relay Round Problems

First Round

1A.Find $\left[\frac{2016!+2013!}{2015!+2014!}\right]$ where $[n]$ represents the largest integer not greater than n.

1B.Let T = TR (The number you will Receive).Let a, b, T, c, and d be five different positive integers that are listed in ascending order from smallest to largest. If their average is T, find the largest possible value for d.

Second Round

2A.Given a sequence $\{a_n\}$ that satisfies $a_1=7$, $a_2=29$, $a_{n+2}=7a_{n+1}-10a_n$ where $n=0, 1, 2, \cdots$. Find the units (last) digit for the term a_{2015}.

2B.Let T = TR. Suppose the volume of the external sphere of a rectangular solid is $8\pi\sqrt{2T}$. Find the maximum possible surface area of this rectangular solid.

Third Round

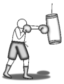
3A.Findthe sum of real roots of equation $2x^3-10x^2+7x+10=0$.

3B.Let T = TR. Given a rectangle with integer value lengths as its dimensions and area of $S=T^2-1$. Find the length of the shortest possible diagonal of this rectangle.

Relay Round Answers

1A.2015.	2A.3.	3A.5.
1B.6041.	2B.48.	3B.$\sqrt{52}$.

Individual Round Problems



First Round

1. Use a certain number of identical cubes of edge length 1 to compose a geometric solid. Among all the solids with front and left side views look like the figure 1, what is the volume of the solid that has the smallest volume?

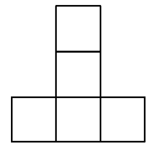


Fig.1

2. Compute $\log_{\sqrt{2}}(2^{\log_7 3}\times 2^{2\log_7 9})^{2\log_3 7}$.

3. Find the minimum value of the function $y=4\cos^2 x-4\cos x-6$.

4. Suppose M is a natural number and $0<M<a<M+1$ where $2^a+a^3=47$. Find M.

5. If $\tan\alpha=\sqrt{2}$, find $\sin 2\alpha-\cos 2\alpha$.

6. Suppose T is a 9-digit natural number, n is a positive natural number, and $3T+1=10^n$. Find the sum of digits of T.

7. If $\sqrt{3}\sin 20^\circ=\cos 40^\circ+\sin x$ and $0^\circ\leqslant x<360^\circ$, find x.

8. Suppose, in $\triangle ABC$, $\cos A=\dfrac{1}{3}$ and $BC=\sqrt{6}$. Find the length of the longest chord of this triangle's circumscribed circle.

9. Given an arithmetic (Equal Difference) sequence $\{a_n\}$ with first term a_1 and common difference d. If $-1\leqslant a_2\leqslant 2$ and $3\leqslant a_5\leqslant 5$, find the range of values of a_{10}.

10. Suppose the lengths of both the edges and main body diagonals of a rectangular solid are natural numbers and this rectangular solid has a volume of 220. Find the length of its main body diagonal.

11. How many real roots does the equation $x^3-x^2-1=0$ have?

12. Suppose 4 rays were drawn from a point in 3-space so that all the angles between any two rays are the same. Find $\cos(\alpha)$ where α is that common angle measurement.

13. The figure 2 consists of nine 1×1 squares. The middle square is divided by a line as indicated in the figure. If a point M is randomly placed inside that middle square, what is the probability of this point M is landed in the shaded region?

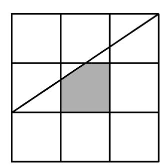


Fig.2

14. Suppose $a+b+c+d=6$ where a, b, c, and d are positive. Find the maximum value for $ab+bc+cd+da$.

Second Round

15. If the solution set for inequality $\sqrt{5-x^2} > ax + \frac{5}{3}$ is $(-2, 1)$, find the value of real number a.

16. Given that $f(x) = \frac{2^{x+2} + 2^{1-x}}{2^x + 2^{-x}}$. If $f(t) = 4$, find the value of $f(-t)$.

17. Given that $y = f(x)$ is a real even function and $f(x-2) = f(x+2)$. Also, $f(x) = x + 2$ when $x \in [-2, 0]$. Find the analytic formula for $f(x)$ when $x \in [2, 6]$.

18. Given that $f(\ln x) = x^2 - 1$. Let $f^{-1}(x)$ be the inverse function of $f(x)$. Find the value of $f^{-1}(3)$.

Third Round

19. Given a sphere O that has diameter $PC = 2$. Let A and B be two points on this sphere such that $AB = \sqrt{2}$ and $\angle APC = \angle BPC = 45^\circ$. Find the volume of the triangular pyramid P - ABC.

20. Given that $a_n = \sqrt{5 \times 6} + \sqrt{7 \times 8} + \cdots + \sqrt{(2n+3)(2n+4)}$ and $b_n = \left[\frac{a_n}{n}\right]$ where $n = 1, 2, 3, \cdots$ ($[x]$ represents the largest integer that is not larger than x). Let $S_n = b_1 + b_2 + b_3 + \cdots + b_n$. Find the sum of all n such that $S_n \leqslant 5455$.

Individual Round Answers

1. 5.
2. 20.
3. -7.
4. 3.
5. $\frac{2\sqrt{2}+1}{3}$.
6. 27 .
7. 190° or 350°.
8. $\frac{3}{2}\sqrt{3}$.
9. $\frac{14}{3} \leqslant a_{10} \leqslant 15$.
10. 15.
11. 1.
12. $-\frac{1}{3}$.
13. $\frac{11}{12}$.
14. 9.
15. $\frac{1}{3}$.
16. 2.
17. $2 - |x - 4|$.
18. $\ln 2$.
19. $\frac{1}{3}$.
20. 5050.