

2014 WMTA Advanced Level

Team Round Problems

1. Define function

$$F(x, y) = |x - 1| + |x - 2| + |x - 3| + |x - 4| + |y - 1| + |y - 2| + |y - 3|.$$

Find the minimum value for $F(x, y)$.

2. Suppose $\mathbf{A} = \{(x, y) \mid y \geq 2x^2\}$ and $\mathbf{B} = \{(x, y) \mid x^2 + (y - a)^2 \leq 5\}$. If $\mathbf{A} \cap \mathbf{B} = \mathbf{B}$, find the range of values for a .
3. Suppose positive integers a and b are relatively primes and when b is divided by a , 4 and 7 are their remainder and quotient, respectively. Let $a_1, a_2, a_3, a_4, \dots$ be all the numbers a (in ascending order) that satisfy the above conditions, find a_{2014} .
4. If the range of values for the function $f(x) = \log_{10} \left[\frac{1}{2}x^2 - (a + 2)x + a^2 + 4 \right]$ is all real numbers, find the domain of $f(x)$.
5. Consider a function $f(x)$ on real numbers \mathbf{R} and satisfies the following conditions:
- (a) $f(2 + x) = f(2 - x)$,
 - (b) $f(4 - x) = -f(4 + x)$, and
 - (c) $f(x) = x^2$ when $0 \leq x \leq 2$.
- Find the value for $f(2015)$.
6. Solve the equation $27^{3x^2+2y} + 27^{3y^2+2z} + 27^{3z^2+2x} = 1$ ($x, y, z \in \mathbf{R}$) for (x, y, z) .
7. Suppose $f(x) = |x^3 - x| - |x^3 + x|$. If the equation $f^2(x) + 2|f(x)| + n - 1 = 0$ ($n \in \mathbf{R}$) has exactly 3 distinct real roots, find the value for n .
8. Find all possible positive integer solutions x and y for $2\sqrt{x+y} - \sqrt{x} - \sqrt{y} = 3$.
9. Let $\{a_n\}$ be a geometric (equal proportion) sequence with $a_n > 0$. Suppose $a_4 a_{2n-4} = 4^n$ ($n \geq 3$) and let S_n be the sum of the first n terms of the sequence $\{\log_2 a_{2n-1}\}$. Find the largest positive integer n that satisfies $S_{2n-1} \leq 2015$.
10. Let $P-ABC$ be a tetrahedron that is inscribed inside sphere O . If $AC = BC = 6$, $\angle ACB = 90^\circ$, and $PB = 12$ is the diameter of sphere O , find the volume of $P-ABC$.
11. Suppose, for any positive integers m and n , function f satisfies the following conditions:
- (a) $f(1, 1) = 2$,
 - (b) $f(m, n + 1) = f(m, n) + (-1)^n \cdot 2$,

(c) $f(m + 1, 1) = (-1)^m \cdot 2f(m, 1)$.

Find the value for $f(2015, 2016)$.

12. If $a, b \geq 0$ and $a + b = 4$, find the range of values for $(a^2 + 2)(b^2 + 2)$.

13. As shown in the Fig.1, all the centers of semi-circles O_1, O_2, O_3, \dots , and O_n are on AC and each semi-circle tangents to its neighbors at points B_1, B_2, B_3, \dots , and B_n .

Suppose CM_1 is also tangent to these semi-circles with points of tangency at M_1, M_2, M_3, \dots , and M_n . If $AB_1 = 2$ and $\angle M_1O_1C = \theta$, find the value of $O_1B_1 + O_2B_2 + O_3B_3 + \dots + O_nB_n$ (in terms of θ) when $n \rightarrow +\infty$.

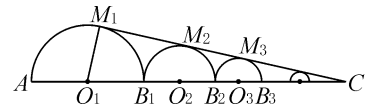


Fig.1

14. If $2 \leq x^2y \leq 4$ and $-2 \leq \frac{3y^2}{x} \leq -1$ where $x, y \in \mathbf{R}$, find the sum of the maximum and minimum values of $\frac{y^4}{x^7}$.

15. Let $\triangle ABC_1$ and $\triangle ABC_2$ be isosceles right triangles both with equal side length of 1. If folding along AB to make the two half planes in forming a dihedral angle of 60° , what is the maximum possible length for C_1C_2 ?

16. Suppose the three different edge lengths of a rectangular box are m, n , and 1. If m and n satisfy $3m + 2n + 6mn = 9m^2 + 4n^2 + 1$, find the length of this rectangular box's main diagonal.

17. Suppose $f(x) = 2\cos^2x - 2\sqrt{3}\sin x \cos x$. If the range of values of $f(x)$ is $[0, 1]$ when $x \in \left[\frac{\pi}{12}, t\right]$, find the value for t .

18. Suppose the sequence $\{a_n\}$ satisfies $a_1 = 0$ and $a_{n+1} = a_n + (n + 1) \cdot 2^{n-1}$. Find the maximum value among all $C_n = \frac{a_n}{3^{n-1}}$.

19. Given a pyramid $P - ABCD$ where base $ABCD$ is a right angle trapezoid, $\angle A = 90^\circ$, $AB \parallel CD$, and $PD \perp$ plane $ABCD$. If $PD = CD = 2$ and $AB = AD = 1$, find the volume of the sphere that is determined by the 4 points P, B, C , and D .

20. Given positive integers a, b , and c such that $a < c$ and $a + c = 2b$. Find the number of 3-digit numbers \overline{abc} that satisfy these conditions.

Team Round Answers

1. 6.
2. $a \geq 10 \frac{1}{8}$.
3. 28221.
4. $\{x \mid x \neq 4\}$.
5. -1 .
6. $(x, y, z) = \left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$.
7. 1.
8. $(x, y) = (16, 9)$ or $(9, 16)$.
9. 22.
10. $36\sqrt{2}$.
11. $-2^{2015} - 2$.
12. $[32, 36]$.
13. $\frac{1 + \cos\theta}{2\cos\theta}$.
14. $-\frac{11}{144}$.
15. $\sqrt{2}$.
16. $\frac{7}{6}$.
17. $\frac{\pi}{6}$.
18. $\frac{8}{9}$.
19. $\frac{8\sqrt{2}}{3}\pi$.
20. 16.

Relay Round Problems



First Round

- 1A. Find the number of 3-digit numbers abc (a , b , and c are distinct) such that $|a - c| = 5$.
- 1B. Let $T = \text{TNYWR}$ (The Number You Will Receive) and $S = \frac{T}{3}$. If $x + y + z = S$, find the maximum value for $xyz + 3(xy + yz + zx)$.



Second Round

- 2A. Suppose $abcdefabc$ is a 9-digit number with $def = 2abc$. If $abcdefabc$ is a product of the squares of 4 distinct prime numbers, find the sum of all possible 3-digit numbers abc .
- 2B. Let $T = \text{TNYWR}$ (The Number You Will Receive). Suppose c is the units digit for T . How many 5-digit numbers in the form of $12abc$ can have 3 as remainder when it is divided by 7?



Third Round

- 3A. Given a sequence $\{a_n\}$ where $a_n = n(n!)$. Let S_n be the sum of the first n terms of $\{a_n\}$. Find the smallest number n so that $S_n \geq 2014$.
(Note: $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$)
- 3B. Let $T = \text{TNYWR}$ (The Number You Will Receive). Find the number of negative value solutions to the equation $-x^5 + x^4 - 3x^3 + 5x^2 - 2x + T = 0$.

Relay Round Answers

- | | | |
|----------|---------|-------|
| 1A.72. | 2A.650. | 3A.6. |
| 1B.1088. | 2B.14. | 3B.0. |

Individual Round Problems



First Round

1. If $ABCD - A'B'C'D'$ is a rectangular solid, how many tetrahedrons can be formed using the center of this rectangular solid as vertex and three points from vertices $A, B, D, B', C',$ and D' to form the base?
2. If $x^5 + 5x^4 + 10x^3 + 10x^2 - 5x + 1 = 0$ ($x \neq -1$), find the value for $(x + 1)^4$.
3. Suppose $x \in \mathbf{R}$ and $[x]$ represents the largest integer not larger than x . If the function $f(x) = \frac{[x]}{x} - a$ ($x > 0$) has exactly three zeros, find the range of possible values for a .
4. Let $A(2, 2)$ be a point on the xy -coordinate system, B a point on the line $y = x + 1$ and C a point on the x -axis. Find the minimum perimeter for all such possible triangles $\triangle ABC$.



Second Round

5. Suppose, except for PA , the length of all edges in tetrahedron $P - ABC$ has a length of 1. Find the range of possible values for the length of PA ?
6. Given that $f(\log_{10}(\log_3 10)) = 5$ for function $f(x) = \frac{ax^3 + b \sin x}{x^2 + c}$ ($a, b, c \in \mathbf{R}$). Find the value for $f(\log_{10}(\log_{10} 3))$.
7. Suppose sets \mathbf{A} and \mathbf{B} are defined to be $\mathbf{A} = \{x, xy, \log_{10}(xy)\}$ and $\mathbf{B} = \{0, |x|, y\}$. If $\mathbf{A} = \mathbf{B}$, find $\left(x + \frac{1}{y}\right) + \left(x^2 + \frac{1}{y^2}\right) + \left(x^3 + \frac{1}{y^3}\right) + \cdots + \left(x^{2014} + \frac{1}{y^{2014}}\right)$.
8. Randomly pick two diagonals (including both face diagonals and body diagonals) from a cube. What is the probability that these two diagonals are perpendicular?



Third Round

9. Given $x^5 + 5 \sin x + 2m = 0$ and $16y^5 + 5 \sin y \cos y - m = 0$ where $m \in \mathbf{R}$ and $x, y \in \left(-\frac{\pi}{6}, \frac{\pi}{6}\right)$. Find the value for $\cos(x + 2y)$.
10. Given a rectangular box $ABCD - A_1B_1C_1D_1$ with the base $ABCD$ being a square and that E and F are points on edges BB_1 and DD_1 , respectively, so that $AB = DF = B_1E = 2$ and $BE = 1$. Find the volume of the pyramid $D_1 - AEC_1F$.

11. Let A , B , and C be the interior angles of triangle $\triangle ABC$ that are opposite to the sides a , b , and c , respectively. If the area of $\triangle ABC$ is $S = \frac{1}{2}bc \cos A = 2\sqrt{2}$ and $a = 2\sqrt{5 - 2\sqrt{2}}$, find the value for $b + c$.
12. Suppose x_i ($i = 1, 2, 3, 4, 5$) are non-negative real numbers and $x_1 + x_2 + x_3 + x_4 + x_5 = 1$. Use $\max\{x, y\}$ to denote the larger of x and y . Find the smallest possible value for $\max\{x_1 + x_2, x_2 + x_3, x_3 + x_4, x_4 + x_5\}$.



Fourth Round

13. Circle C , x -axis, y -axis, and the curve $y = \frac{3}{x}$ ($x > 0$) are tangent to each other as shown in the Fig.1. Find the radius of circle C .

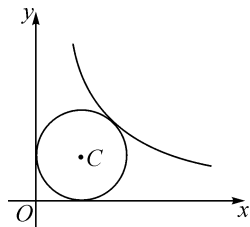


Fig.1

14. Find the area of the region that is bounded by the curve $|x - 2| - |y + 1| = |2x - 7|$.



Fifth Round

15. As shown in the Fig.2, A_1, A_2, A_3 , and A_4 are points on the x -axis and B_1, B_2, B_3 , and B_4 are points on the curve $y^2 = kx$ ($k > 0$). Suppose points C_1, C_2 , and C_3 are points on A_2B_2, A_3B_3 , and A_4B_4 , respectively, so that $A_1B_1C_1A_2, A_2B_2C_2A_3$, and $A_3B_3C_3A_4$ are all squares with areas, respectively, S_1, S_2 , and S_3 . If $OA_1 = 1$ and $S_2 = 2S_1$, find S_3 . (The result cannot contain k).

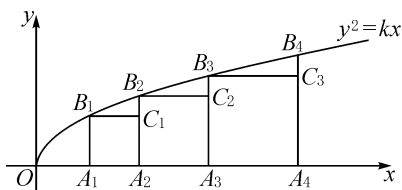


Fig.2

16. Consider the sequence $\{a_n\}$ with $a_1 = \frac{1}{4}$. Denote the sum of its first n terms as S_n . If a_n is the arithmetic average of $\sqrt{S_n}$ and $\sqrt{S_{n-1}}$ for all $n \geq 2$, find the value for a_{2014} .

Individual Round Answers

- | | | | |
|--|---------------------|---------------------|------------------------------|
| 1.8. | 6. -5 . | 10. $\frac{4}{3}$. | 13. $2\sqrt{3} - \sqrt{6}$. |
| 2.10. | 7.0. | 11.6. | 14.3. |
| 3. $\left(\frac{3}{4}, \frac{4}{5}\right]$. | 8. $\frac{3}{10}$. | 12. $\frac{1}{3}$. | 15. $2 + \sqrt{2}$. |
| 4. $\sqrt{26}$. | 9.1. | | 16. $1006 \frac{3}{4}$. |
| 5. $(0, \sqrt{3})$. | | | |